

# SOME RECENT RESULTS IN CALCULATION OF THE CASIMIR ENERGY AT ZERO AND FINITE TEMPERATURE

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The survey summarizes briefly the results obtained recently in the Casimir effect studies considering the following subjects: i) account of the material characteristics of the media and their influence on the vacuum energy (for example, dilute dielectric ball); ii) application of the spectral geometry methods for investigating the vacuum energy of quantized fields with the goal to gain some insight, specifically, in the geometrical origin of the divergences that enter the vacuum energy and to develop the relevant renormalization procedure; iii) universal method for calculating the high temperature dependence of the Casimir energy in terms of heat kernel coefficients.

## 1. Casimir energy of a dilute dielectric ball

Calculation of the Casimir energy of a dielectric ball has a rather long history starting 20 years ago.<sup>1</sup> However only recently the final result was obtained for a dilute dielectric ball at zero<sup>2,3</sup> and finite<sup>4,5</sup> temperature. Here we summarize briefly the derivation of the Casimir energy of a dilute dielectric ball by making use of the mode summation method and the addition theorem for the Bessel functions instead of the uniform asymptotic expansion for these functions.<sup>3,4</sup>

A solid ball of radius  $a$  placed in an unbounded uniform medium is considered. The contour integration technique<sup>3</sup> gives ultimately the following

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representation for the Casimir energy of the ball

$$E = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l+1) \int_0^{y_0} dy y \frac{d}{dy} \ln \left[ W_l^2(n_1 y, n_2 y) - \frac{\Delta n^2}{4} P_l^2(n_1 y, n_2 y) \right], \quad (1)$$

where

$$\begin{aligned} W_l(n_1 y, n_2 y) &= s_l(n_1 y) e'_l(n_2 y) - s'_l(n_1 y) e_l(n_2 y), \\ P_l(n_1 y, n_2 y) &= s_l(n_1 y) e'_l(n_2 y) + s'_l(n_1 y) e_l(n_2 y), \end{aligned}$$

and  $s_l(x)$ ,  $e_l(x)$  are the modified Riccati-Bessel functions,  $n_1, n_2$  are the refractive indices of the ball and of its surroundings,  $\Delta n = n_1 - n_2$ .

Analysis of divergences<sup>3</sup> leads to the following algorithm for calculating the vacuum energy (1) in the  $\Delta n^2$ -approximation. First, the  $\Delta n^2$ -contribution should be found, which is given by the sum  $\sum_l W_l^2$ . Upon changing its sign to the opposite one, we obtain the contribution generated by  $W_l^2$ , when this function is in the argument of the logarithm. The  $P_l^2$ -contribution into the vacuum energy is taken into account by expansion of Eq. (1) in terms of  $\Delta n^2$ .

Applying the addition theorem for the Bessel functions

$$\sum_{l=0}^{\infty} (2l+1) [s'_l(\lambda r) e_l(\lambda \rho)]^2 = \frac{1}{2r\rho} \int_{r-\rho}^{r+\rho} \left( \frac{1}{\lambda} \frac{\partial G}{\partial r} \right)^2 R dR$$

with

$$G = \lambda r \rho R^{-1} e^{-\lambda R}, \quad R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}$$

one arrives at the result

$$E = \frac{23}{384} \frac{\Delta n^2}{\pi a} = \frac{23}{1536} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\pi a}, \quad \varepsilon_i = n_i^2, \quad i = 1, 2.$$

Extension to finite temperature  $T$  is accomplished by substituting the  $y$ -integration in (1) by summation over the Matsubara frequencies  $\omega_n = 2\pi n T$ .

In the  $\Delta^2$ -approximation the last term in Eq. (3.20) from the article<sup>4</sup>

$$\overline{U}_W(T) = 2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 \int_{\Delta n}^2 \frac{e^{-2w_n R}}{R} dR, \quad w_n = 2\pi n a T \quad (2)$$

can be represented in the following form

$$\overline{U}_W(T) = -2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 E_1(4w_n), \quad (3)$$

where  $E_1(x)$  is the exponential-integral function. Now we accomplish the summation over the Matsubara frequencies by making use of the Abel-Plana formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x) dx + i \int_0^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} dx. \quad (4)$$

The first term in the right-hand side of this equation gives the contribution independent of the temperature, and the net temperature dependence is produced by the second term in this formula. Being interested in the low temperature behavior of the internal energy we substitute into the second term in Eq. (4) the following expansion of the function  $E_1(z)$

$$E_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k \cdot k!}, \quad |\arg z| < \pi, \quad (5)$$

where  $\gamma$  is the Euler constant. The contribution proportional to  $T^3$  is produced by the logarithmic term in the expansion (5). The higher powers of  $T$  are generated by the respective terms in the sum over  $k$  in this formula ( $t = 2\pi aT$ )

$$\bar{U}_W(T) = \frac{\Delta n^2}{\pi a} \left( -\frac{1}{96} + \frac{\zeta(3)}{4\pi^2} t^3 - \frac{1}{30} t^4 + \frac{8}{567} t^6 - \frac{8}{1125} t^8 + O(t^{10}) \right). \quad (6)$$

Taking all this into account we arrive at the following low temperature behavior of the internal Casimir energy of a dilute dielectric ball

$$U(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} + \frac{\zeta(3)}{4\pi^2} t^3 - \frac{7}{360} t^4 + \frac{22}{2835} t^6 - \frac{46}{7875} t^8 + O(t^{10}) \right). \quad (7)$$

The relevant thermodynamic relation give the following low temperature expansions for free energy

$$F(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} - \frac{\zeta(3)}{8\pi^2} t^3 + \frac{7}{1080} t^4 - \frac{22}{14175} t^6 + O(t^8) \right). \quad (8)$$

For large temperature  $T$  we found<sup>4</sup>

$$U(T) \simeq \Delta n^2 T / 8, \quad F(T) \simeq -\Delta n^2 T [\ln(aT) - c] / 8, \quad (9)$$

where  $c$  is a constant<sup>5,6</sup>  $c = \ln 4 + \gamma - 7/8$ . Analysis of Eqs. (3.20) and (3.31) from the paper<sup>5</sup> shows that there are only exponentially suppressed corrections to the leading terms (9).

Summarizing we conclude that now there is a complete agreement between the results of calculation of the Casimir thermodynamic functions

for a dilute dielectric ball carried out in the framework of two different approaches: by the mode summation method<sup>3,4</sup> and by perturbation theory for quantized electromagnetic field, when dielectric ball is considered as a perturbation in unbounded continuous surroundings.<sup>5</sup>

## 2. Spectral geometry and vacuum energy

In spite of a quite long history of the Casimir effect (more than 50 years) deep understanding and physical intuition in this field are still lacking. The main problem here is the separation of net finite effect from the divergences inevitably present in the Casimir calculations. A convenient analysis of these divergences is provided by the heat kernel technique, namely, the coefficients of the asymptotic expansion of the heat kernel.

Keeping in mind the elucidation of the origin of these divergences in paper<sup>7</sup> the vacuum energy of electromagnetic field has been calculated for a semi-circular infinite cylindrical shell. This shell is obtained by crossing an infinite cylinder by a plane passing through its symmetry axes. In the theory of waveguides it is well known that a semi-circular waveguide has the same eigenfrequencies as the cylindrical one but without degeneracy (without doubling) and safe for one frequency series. Notwithstanding the very close spectra, the vacuum divergences in these problems prove to be drastically different, so the zeta function technique does not give a finite result for a semi-circular cylinder unlike for a circular one.

It was revealed that the origin of these divergences is the corners in the boundary of semi-circular cylinder.<sup>8</sup> In terms of the heat kernel coefficients, it implies that the coefficient  $a_2$  for a semi-circular cylinder does not vanish due to these corners.

However in the 2-dimensional (plane) version of these problems the origin of nonvanishing  $a_2$  coefficient for a semicircle is the contribution due to the curvature of the boundary, while the corner contributions to  $a_2$  in 2 dimensions are cancelled.

Different geometrical origins of the vacuum divergences in the two- and three-dimensional versions of the boundary value problem in question evidently imply the impossibility of obtaining a finite and unique value of the Casimir energy by taking advantage of the atomic structure of the boundary or its quantum fluctuations. It is clear, because any physical reason of the divergences should hold simultaneously in the two- and three-dimensional versions of a given boundary configuration.

### 3. High temperature asymptotics of vacuum energy in terms of heat kernel coefficients

The Casimir calculations at finite temperature prove to be a nontrivial problem specifically for boundary conditions with nonzero curvature. For this goal a powerful method of the zeta function technique and the heat kernel expansion can be used. For obtaining the high temperature asymptotics of the thermodynamic characteristics it is sufficient to know the heat kernel coefficients and the determinant for the spatial part of the operator governing the field dynamics. This is an essential merit of this approach.<sup>6</sup>

Starting point is the general high temperature expansion of the free energy in terms of the heat kernel coefficients.<sup>9</sup> These coefficients needed for are calculated as the residua of the corresponding zeta functions.<sup>10</sup>

#### 3.1. *A perfectly conducting spherical shell in vacuum*

The first six heat kernel coefficients in this problem are:

$$a_0 = a_{1/2} = a_1 = 0, \quad \frac{a_{3/2}}{(4\pi)^{3/2}} = \frac{1}{4}, \quad a_2 = 0, \quad \frac{a_{5/2}}{(4\pi)^{3/2}} = \frac{c^2}{160 R^2}. \quad (10)$$

Furthermore

$$a_j = 0, \quad j = 3, 4, 5, \dots \quad (11)$$

The exact value of  $\zeta'(0)$  is derived in Ref.<sup>6</sup>

$$\zeta'(0) = 0.38429 + (1/2) \ln(R/c). \quad (12)$$

As a result we have the following high temperature limit for the free energy

$$F(T) = -\frac{T}{4} \left( 0.76858 + \ln \tau + \frac{1}{960\tau^2} \right) + O(T^{-3}), \quad (13)$$

where  $\tau = RT/(\hbar c)$  is the dimensionless ‘temperature’. The expression (13) exactly reproduces the asymptotics obtained in Ref.<sup>11</sup> by making use of the multiple scattering technique (see Eq. (8.39) in that paper).

#### 3.2. *A perfectly conducting cylindrical shell*

The heat kernel coefficients are

$$a_0 = a_{1/2} = a_1 = a_2 = 0, \quad \frac{a_{3/2}}{(4\pi)^{3/2}} = \frac{3}{64 R}, \quad \frac{a_{5/2}}{(4\pi)^{3/2}} = \frac{153}{8192} \frac{c^2}{R^3}. \quad (14)$$

The zeta function determinant in this problem is calculated in Ref.<sup>6</sup>

$$\zeta'(0) = 0.45711/R + (3/32 R) \ln(R/2c). \quad (15)$$

The free energy behavior at high temperature is the following

$$F(T) = -\frac{T}{R} \left( 0.22856 + \frac{3}{64} \ln \frac{\tau}{2} - \frac{51}{65536\tau^2} \right) + O(T^{-3}). \quad (16)$$

The high temperature asymptotics of the electromagnetic free energy in presence of perfectly conducting cylindrical shell was investigated in paper.<sup>11</sup> To make the comparison handy let us rewrite their result as follows

$$F(T) \simeq -(T/R) [0.10362 + (3/64R) \ln(\tau/2)]. \quad (17)$$

The discrepancy between the terms linear in  $T$  in Eqs. (16) and (17) is due to the double scattering approximation used in Ref.<sup>11</sup> Our approach gives the exact value of this term (see Eq. (16)).

#### 4. Conclusion

The inferences concerning the individual subjects of this review have been done in respective sections. Here we only note, that in order to cast the theory of the Casimir effect to a complete form further studies are certainly needed.

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